arXiv:hep-th/9706011v2 5 Nov 1998

Reduction of internal degrees of freedom in the large N limit in matrix models[*]

Óscar Diego[†]

The Niels Bohr Institute
Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark

Abstract

In this paper the large N limit of one hermitian matrix models coupled to an external matrix is considered. It is shown that in the large N limit the number of degrees of freedom are reduced to be order N even though it is order N^2 for finite N. It is claimed that this result is the origin of the factorization of observables in the path integral formalism.

[†]e-mail: diego@nbivms.nbi.dk

1. Introduction

It is a well known fact that in the large N limit the vacuum expectation value of a finite product of observables is the product of vacuum expectation values. It is very easy, almost trivial, to prove this remarkable factorization in the Feynman diagram approach[1]. It is also possible to prove it in the canonical formalism[2]. But in the path integral formalism it is not clear how this factorization holds. This factorization suggests that only one configuration enter into the path integral. This is true for vector models but in matrix models the number of degrees of freedom is order N^2 and the expansion parameter of the large N limit is $1/N^2$, therefore the quantum corrections to any classical configuration must be order one and fluctuations to all orders must enter into the path integral. This explains why is so difficult to find the master field: because it is not given by one configuration but arises from several non equivalent configurations. For instance, in one hermitian matrix models the partition function is given by

$$Z = \int D\Phi \exp\{-Ntr[V(\Phi)]\}. \tag{1}$$

The potential is a function of N degrees of freedom only: the eigenvalues of Φ. Actually

$$Z = \int DU D\Lambda \Delta^2 \exp\{-Ntr[V(\Lambda)]\} \tag{2}$$

where Λ is the diagonal matrix of the eigenvalues of Φ, Δ is a Van der Monde determinant and the integration over the set of unitary matrices DU is trivial. The leading contribution to the partition function is given by only one configuration of eigenvalues, the master field, which is a solution of the

saddle-point condition

$$\frac{\partial}{\partial \Lambda_{ii}} \left(\ln\{\Delta^2 \exp\{-Ntr[V(\lambda)]\}\} \right) = 0 \qquad (3)$$

In this case the factorization of observables holds trivially.

But if we add to the matrix potential a coupling with an external matrix A then the potential becomes a function of N^2 degrees of freedom. Actually

$$Z = \int DU D\Lambda \Delta^2 \exp\{-Ntr[V(\Lambda) + U^\dagger \Lambda U A]\} \qquad (4)$$

We can prove the factorization of a product of observables which are functions of the eigenvalues, like $< Tr\Phi Tr\Phi >$. After the exact integration over DU the partition function becomes

$$Z(A) \propto \int D\Lambda \Delta \exp\{-Ntr[V(\Lambda) + \Lambda A]\} \qquad (5)$$

Only one configuration of eigenvalues yields the leading contribution to the partition function in the large N limit. Therefore we can also define a master field in this case but first we must perform the integration over N^2 degrees of freedom exactly, in other words all configurations of the unitary matrix variable U enter into the master field.

The existence of a master field for the eigenvalues explains the factorization of observables like $< Tr\Phi Tr\Phi >$. But for observables like

$$< Tr(\Phi A)Tr(\Phi A) >=< Tr(U^\dagger \Lambda U A)Tr(U^\dagger \Lambda U A) > \qquad (6)$$

the factorization is difficult to understand. Of course we know that the factorization holds, we can prove it perturbatively, but we do not understand the factorization of observables nonperturbatively in the path integral approach.

We can expect that in the large N limit the partition function is given by the integral representation of finite N but restricted over some subset of configurations. For instance (4) in the large N limit should be

$$Z = \int_{\Gamma_\Lambda} D\lambda \int_{\Gamma_U} DU \Delta^2 \exp\{-Ntr[V(\Lambda) + U^\dagger \Lambda U A]\} \qquad (7)$$

where Γ_λ and Γ_U are subsets of the set of all eigenvalues and of all unitary matrices respectively. We know that there is only one configuration in the subset Γ_λ. The factorization

$$< Tr\Phi A Tr\Phi A >=< Tr\Phi A >< Tr\Phi A > \qquad (8)$$

suggests that there is only one configuration in Γ_U. But at first sight this is not possible because the set of unitary matrices is labeled by N^2 degrees of freedom. But if there are several nonequivalent configurations in Γ_U the factorization of observables like (8) becomes very unnatural. This old problem has been pointed out by Itzykson and Zuber in[3] several years ago.

In order to understand this problem we must find the configurations which enter into the path integral in the large N limit. In this paper we will study one hermitian matrix models coupled to an external matrix. Actually we will study:

$$\begin{aligned} Z &= \int D\Phi \exp\{-NTr[V(\Phi) + A\Phi]\} \\ Z &= \int D\Phi \exp\{-NTr[V(\Phi) + A\Phi A\Phi]\} \end{aligned} \qquad (9)$$

The first model has been solved in[3, 4] and the second in[7]. But we can not read from the solutions of the above models the configurations that yield the leading contribution to the large N limit. In the usual approaches one

3

must perform an integration over a subset of variables exactly. In order to understand the factorization of observables we must study the large N limit before performing any integration.

We will show that, in the above matrix models, the configurations which enter into the path integral in the large N limit can be labeled with only N variables. For instance the subset Γ_U is a subset of dimension N. Therefore the quantum corrections to any classical configuration are subleading in the large N limit. Hence only one configuration yields the leading contribution to the path integral in the large N limit and the factorization of observables follows trivially.

2. Integration over the unitary matrix in the large N limit

Let us start with the following matrix model

$$Z(A) = \int D\Phi \exp\left\{-N[Tr(\Phi A) + TrV(\Phi)]\right\}. \tag{10}$$

Without lost of generality we can consider diagonal external matrices.

Let us perform the change of variables

$$\Phi \rightarrow U^\dagger \Lambda U, \tag{11}$$

where Λ is the diagonal matrix of the eigenvalues of Φ and U is a unitary matrix. The partition function becomes:

$$Z(A) = \int D\Lambda \Delta^2 \exp\{-NTrV(\Lambda)\}\tilde{Z}(A,\Lambda), \tag{12}$$

where

$$\tilde{Z}(A,\Lambda) = \int DU \exp\{-NTr(U^\dagger \Lambda U A)\}. \tag{13}$$

This integral can be solved with the Itzykson-Zuber formula[3]. But we will show that in the large N limit the above integral with Λ fixed can be reduced

4

to an integral over unitary matrices with only order N nonvanishing elements. Hence, as in vector models, we can use saddle-point methods to calculate it.

In the introduction I have claimed that at first sight an integral like (13) cannot be solved with a saddle-point method. This is because any correction must be order the number of degrees of freedom which, in matrix models, is the same as the inverse of the expansion parameter. Therefore any correction must be order one. But the above argument does not take into account the compactness of the unitary set. Unitary matrices must satisfy the constraint

$$UU^\dagger = I \tag{14}$$

Therefore, the absolute values of the elements of an arbitrary unitary matrix must decrease when N increase. Hence the range of variation of U decrease with N. But there are exceptions to this rule. For instance the absolute values of the non zero elements of a diagonal matrix are order one no matter how bigger is N. Actually for bidiagonal, tridiagonal and in general matrices with a finite number of nonzero elements in each row and column, the absolute values of its non zero elements are order one. Therefore we can expect that these configurations must play some special role in the large N limit. Actually I will show that these configurations yield the leading contribution to the partition function in the large N limit.

There are two kind of unitary matrices. Unitary matrices with order $N^2 - N$ elements so smaller that they decoupled from the matrix potential in the large N limit, let us call them unitary matrices with N degrees of freedom, and unitary matrices where order N^2 elements will be coupled between them through the matrix potential in the large N limit, let us call them unitary matrices with N^2 degrees of freedom. Hence the partition function should

be the sum of two terms

$$Z = Z_{uncoupled} + Z_{coupled} \tag{15}$$

where $Z_{uncoupled}$ is the partition function restricted over the subset of unitary matrices with N degrees of freedom and $Z_{coupled}$ is restricted over the subset of unitary matrices with N^2 degrees of freedom.

It is not possible to solve $Z_{coupled}$ but it is trivial that

$$Z_{coupled} \leq Vol \exp\{-NTr(U_S^\dagger \Lambda U_S A)\} \tag{16}$$

where Vol is the volume of the subset of unitary matrices with N^2 degrees of freedom and U_S is the absolute minimum of the potential in the subset of unitary matrices with N^2 degrees of freedom.

The partition function $Z_{uncoupled}$ is given by

$$Z_{uncoupled} = A \exp\{-N^2 F\} \tag{17}$$

where A is the factor which arises after the integration over the $N^2 - N$ degrees of freedom which decoupled from the potential and I will show that F is the free energy of a matrix model restricted over unitary matrices with only N nonzero elements. I will show that A and Vol are of the same order in the large N limit: this is because the compactness of the unitary group. Then if

$$F < \frac{1}{N} Tr(U^\dagger \Lambda U_S A) \tag{18}$$

then $Z_{uncoupled}$ gives the leading contribution to Z. And because $Z_{uncoupled}$ can be reduced to an integral over N degrees of freedom only; actually I will show that

$$Z_{uncoupled} = \int DU \exp\{-NTr(U^\dagger \Lambda U A)\} \tag{19}$$

6

where DU is the measure over unitary matrices with only N nonzero elements; there is a master field in the variables U and corrections to this master field will be subleading in the large N limit.

In order to classify the unitary matrices according to the absolute values of its elements we must transform the measure over the holomorphic elements

$$DU = \prod_{ab} dU_{ab} \tag{20}$$

into a measure which includes the absolute values of the elements

$$DU = \prod_{a \leq b} d|U_{ab}| \cdots \tag{21}$$

The invariant measure DU can be extended over the set of all matrices as follows:

$$\int DU = \int \prod_{a,b=1}^{N} \left\{ dU_{ab} dU_{ab}^* \delta \left(\sum_{c=1}^{N} U_{ac} U_{bc}^* - \delta_{ab} \right) \right\}. \tag{22}$$

This measure is invariant under the unitary transformation.

Let us perform the following change of variables:

$$(U_{ab}, U_{ab}^*) \rightarrow (|U_{ab}|, \theta_{ab}), \tag{23}$$

where θ_{ab} are the phases of the elements of the unitary matrix. The Jacobian of (23) is

$$\prod_{a,b=1}^{N} |U_{ab}|. \tag{24}$$

Therefore the measure (22) becomes

$$\int DU = \int \prod_{a,b=1}^{N} \left\{ d|U_{ab}| \; d\theta_{ab} \; |U_{ab}| \; \delta \left(\sum_{c=1}^{N} |U_{ac}||U_{bc}| \exp(i\theta_{ac} - i\theta_{bc}) - \delta_{ab} \right) \right\} \tag{25}$$

and the partition function (13) becomes:

$$
\begin{aligned}
\tilde{Z}(A, \Lambda) \;=\; & \int \prod_{a,b=1}^{N} (d|U_{ab}|\, d\theta_{ab}) \prod_{a,b=1}^{N} |U_{ab}| \\
& \prod_{a,b=1}^{N} \delta \left[\sum_{c=1}^{N} |U_{ac}||U_{bc}| \exp(i\theta_{ac} - i\theta_{bc}) - \delta_{ab} \right] \\
& \exp \left\{ -N \sum_{a,b=1}^{N} |U_{ab}|^2 \Lambda_{bb} A_{aa} \right\} .
\end{aligned}
\tag{26}
$$

The variables $|U_{ab}|$ are restricted by the delta functions which yield the following constraints:

$$
\sum_{c=1}^{N} |U_{ac}||U_{bc}| \exp\left(i\theta_{ac} - i\theta_{bc}\right) = \delta_{ab}.
\tag{27}
$$

Let us study now the behaviour of the measure (25) in the large N limit for differents subset of unitary matrices. For homogeneous configurations: all $|U_{ab}|$ are of the same order, the unitary constraints (27) restrict the variables $\{|U_{ab}|\}$ to be order $1/\sqrt{N}$.

Let Z_H the partition function (26) restricted over the set of homogeneous configurations. Then there is a trivial bound to Z_H given by

$$
Z_H \leq Vol \exp \left\{ -NTr(U_s^\dagger \Lambda U_s A) \right\},
\tag{28}
$$

where U_s is the absolute minimum of $Tr(U_s^\dagger \Lambda U_s A)$ in the subset of homogeneous configuration and Vol is the volume of the subset of homogeneous configurations. Because the dimension of the subset of homogeneous configurations is N^2 the volume must be order

$$
Vol \propto (\frac{1}{\sqrt{N}})^{N^2}
\tag{29}
$$

8

Let us study now the partition function Z_I restricted over the following inhomogeneous configurations: for each row and column there are a finite number of elements with absolute value order one, let us call them large elements, and the others are order less than $1/\sqrt{N}$, let us call them small.

Order less than $1/\sqrt{N}$ means that the small elements are order

$$|U_{ab}^{small}| \propto \left(\frac{1}{\sqrt{N}}\right)^{(1+\epsilon)} \tag{30}$$

where ϵ is a positive number which goes to zero in the large N limit.

The are three contribution to the constraints (27). The contribution which comes from the large elements

$$\sum_c |U_{ac}^{large}||U_{bc}^{large}| \exp\left(i\theta_{ac} - i\theta_{bc}\right). \tag{31}$$

This is order one because the sum over c can only have a finite number of terms. Let us remark that these inhomogeneous matrices must have only a finite number of large elements in each row a column, otherwise the above contribution becomes order greater than one for $a = b$ and the matrices are not unitary.

The contribution which comes from the small elements

$$\sum_c |U_{ac}^{small}||U_{bc}^{small}| \exp\left(i\theta_{ac} - i\theta_{bc}\right). \tag{32}$$

This is order

$$\left(\frac{1}{N}\right)^\epsilon \tag{33}$$

because now the sums over c runs over order N elements. And the contribution which comes from the crossing of large and small elements.

$$\sum_c |U_{ac}^{large}||U_{bc}^{small}| \exp\left(i\theta_{ac} - i\theta_{bc}\right). \tag{34}$$

9

This is order

$$\frac{1}{\sqrt{N}} \left(\frac{1}{\sqrt{N}} \right)^{\epsilon} \tag{35}$$

because the sum over c is over a finite number of terms. There are only a finite number of nonzero large elements $|U_{ac}^{large}|$ for fixed a.

Now let us define ϵ as

$$\epsilon = \frac{1}{\sqrt{\ln N}} \tag{36}$$

Then ϵ goes to zero in the large N limit and

$$\left(\frac{1}{\sqrt{N}} \right)^{\epsilon} \to 0 \tag{37}$$

Hence, in the large N limit the unitary constraints becomes

$$\sum_c |U_{ac}^{large}||U_{bc}^{large}| \exp\left(i\theta_{ac} - i\theta_{bc}\right) = \delta_{ab}. \tag{38}$$

Therefore the small elements have decoupled from the unitary constraints. Let us remark that the large elements can be put on a matrix, say U_L, and the constraints (38) are equivalent to the unitarity of U_L.

But we cannot perform the large N limit inside the Dirac delta function, in other words is not true that in the large N limit

$$\delta(UU^\dagger - I) \to \delta(U_L U_L^\dagger - I) \tag{39}$$

This is because the matrix U_L, the matrix of the large elements only, has only order N variables and there are N^2 delta functions in the partition function (26).

Let us consider the unitary constraints up to the first correction in the large N limit. Then

$$\sum_c |U_{ac}^{large}||U_{bc}^{large}| \exp\left(i\theta_{ac} - i\theta_{bc}\right) + \sum_c |U_{ac}^{small}||U_{bc}^{small}| \exp\left(i\theta_{ac} - i\theta_{bc}\right) = \delta_{ab}. \tag{40}$$

The cross term (34) gives the second correction. Because there are only N variables $|U_{ab}^{large}|$, a finite number in each row and column, the large variables only appear in order N constraints. For instance a matrix with N large variables in only one row gives contribution to all the constraints but such matrices are not unitary.

Therefore, there are N delta functions which depend on the large variables and in the large N limit they become.

$$\prod_{(a,b)\in\Gamma} \delta\left(\sum_c |U_{ac}^{large}||U_{bc}^{large}|\exp\left(i\theta_{ac}-i\theta_{bc}\right)-\delta_{ab}\right) \qquad (41)$$

If we call I_N the set given by the first N integers then Γ is a subset of the set $I_N \times I_N$ and has only order N elements. In other words (41) is the product of order N delta functions.

And there are $N^2 - N$ delta functions which are independent of the large variables:

$$\prod_{(a,b)\in\bar{\Gamma}} \delta\left(\sum_c |U_{ac}^{small}||U_{bc}^{small}|\exp\left(i\theta_{ac}-i\theta_{bc}\right)\right) \qquad (42)$$

where $\bar{\Gamma}$ is the complementary of Γ in $I_N \times I_N$. Let us remark that there is a complete decoupling between the large and the small variables in the measure.

Let us define δU as the matrix given by the small elements only. Then the delta functions (41) and (42) means that

$$
\begin{aligned}
(U_L)_{ac}(U_L^\dagger)_{cb} &= \delta_{ab} \ \ (a,b)\in\Gamma \\
(\delta U)_{ac}(\delta U^\dagger)_{cb} &= 0 \ \ (a,b)\in\bar{\Gamma}
\end{aligned}
\qquad (43)
$$

Let us remark that the second equation in (43) does not mean that δU is singular because the indices a and b do not run over all integers between

11

one and N. Actually we can write the second equation as

$$\delta U \delta U^\dagger = J \tag{44}$$

where J is an arbitrary matrix with nonzero elements J_{ab} for the labels (a, b) that belongs to Γ and

$$J_{ab} \propto \left(\frac{1}{N}\right)^\epsilon \quad (a, b) \in \Gamma. \tag{45}$$

Let us consider the matrix potential now. The matrix potential can be split into two parts:

$$V = -N \sum_{a=1}^{N} \sum_{b \in large} |U_{ab}|^2 \Lambda_{bb} A_{aa} - N \sum_{a=1}^{N} \sum_{b \in small} |U_{ab}|^2 \Lambda_{bb} A_{aa} \tag{46}$$

The first term is order N^2 because the sum is extended over the variables which are order one and the sum over the index b has only a finite number of terms. In the second term each sum over indices a and b is order N, but the order of the absolute values are less than $1/\sqrt{N}$ and therefore the second term is order less than N^2. Therefore the matrix potential is a function of the large variables only in the large N limit.

Let us perform the integration over the small variables. We must take into account that

$$
\begin{aligned}
\delta(\delta U \delta U^\dagger - J) &= \frac{1}{\det U} \delta(\delta U^\dagger - J(\delta U)^{-1}) \\
&= \frac{1}{\sqrt{\det J}} \delta(\delta U^\dagger - J(\delta U)^{-1})
\end{aligned}
\tag{47}
$$

Hence, taken into account the decoupling of the small variables, equations (47) and (30), and the definition of ϵ, the integration over the small variables becomes

$$\int D(\delta U) D(\delta U^\dagger) \delta(\delta U \delta U^\dagger - J) \propto \left(\frac{1}{\sqrt{N}}\right)^{((N^2-N)(1+\epsilon)-\epsilon N)} \rightarrow \left(\frac{1}{\sqrt{N}}\right)^{N^2} \tag{48}$$

12

Hence after the integration over the small variables, Z_I becomes in the large N limit

$$Z_I \propto \left(\frac{1}{\sqrt{N}}\right)^{N^2} \int DU_L DU_L^\dagger \delta(U_L^\dagger U_L - 1) \exp\left\{-NTr(U_L^\dagger \Lambda U_L A)\right\} \quad (49)$$

and after the integration over $DU_L^\dagger$

$$Z_I \propto \left(\frac{1}{\sqrt{N}}\right)^{N^2} \int DU_L \exp\left\{-NTr(U_L^\dagger \Lambda U_L A)\right\} \quad (50)$$

where now DU_L is the measure over unitary matrices with only order N elements nonzero. Let us remark that that the number of nonzero elements is not fixed but it is order N. Therefore the dimension of the set of inhomogeneous matrices is not a fixed number. This means that DU_L is not a Riemanian measure in the usual sense. This is not important in the large N limit. Actually I will show that the integral over DU_L is given by only one configuration in the large N limit.

But we can also define DU_L explicitly as a sum

$$DU_L = D_1 U_L + D_2 U_L + \cdots \quad (51)$$

where $D_1 U_L$ is the measure over unitary diagonal matrices, $D_2 U_L$ is the measure over unitary matrices with two diagonals a so on. For instance let us consider the set of inhomogeneous matrices with large N elements in the main diagonal only. Then we perform all the tricks describe above and we arrive to the partition function restricted over inhomogeneous matrices with large elements in the main diagonal

$$Z_I^1 \propto \left(\frac{1}{\sqrt{N}}\right)^{N^2} \int D_1 U_L \exp\left\{-NTr(U_L^\dagger \Lambda U_L A)\right\} \quad (52)$$

13

where D_1U_L is the measure over diagonal unitary matrices. In this way we can write the partition function Z_I as a sum over partition function restricted over diagonal, bidiagonal and so on. The factors in front of each partition function Z_I^j are differents, but all are of the same order in the large N limit. These factors are not important because in the large N limit only one configurations enter into the path integral Z_I.

The partition function Z_I can be solved with a saddle-point method because depends only on order N degrees of freedom. Actually the saddle-point equation is given by

$$\frac{\partial}{\partial U_{ab}}Tr(U^\dagger \Lambda U A) = 0 \tag{53}$$

In this case any correction to the saddle-point solutions are order the expansion parameter, which is $1/N^2$, times the number of degrees of freedom, which are order N. Let us remark that if we perform any change of variables in (50), for instance

$$(U_L)_{ab} \rightarrow (|(U_L)_{ab}|, \theta_{ab}), \tag{54}$$

the saddle-point conditions (53) do not change. This is because the Jacobian of any change of variables grows as an exponential of N, because there are only N degrees of freedom, for instance the jacobian of (54) is the product of N elements

$$\prod_{(a,b)\in\Gamma} |(U_L)_{ab}|, \tag{55}$$

whereas the potential grows as the exponential of N^2.

Taken into account that $U_L U_L^\dagger = I$, the saddle point configuration of (53) are given by:

$$U\Lambda U^\dagger A = AU\Lambda U^\dagger, \tag{56}$$

14

or in other words the matrix Φ must commute with the external matrix A. Actually (56) is the condition of commutativity of Φ and A in the basis where the matrix A is diagonal.

Let us remark that the above equation (56) comes from the fact that U_L is a unitary matrix with only order N nonzero elements. It is not important if all its nonzero elements are order one or $1/\sqrt{N}$. Of course a unitary matrix with only order N nonzero elements must have N large elements.

The same analysis can be carried out for configurations with large elements, small elements and order $1/\sqrt{N}$ elements. For instance configurations with order N large elements and order $N^2 - N$ elements order $1/\sqrt{N}$. Let us remark that there is a very important difference between elements order $1/\sqrt{N}$ and small elements, which are order less than $1/\sqrt{N}$. The former are coupled with the large elements and between them through the potential and the small elements do not enter into the potential and do not coupled with the large elements through the unitary constraint. Nevertheless we must be careful with the above division between large, small and homogeneous elements. An element order $1/\sqrt{N}$ enter into the potential only if there are order N^2 homogeneous elements in the same row or column.

Therefore we can construct configurations with order N^2 degrees of freedom coupled between them starting with the homogeneous configurations and changing a finite number of homogeneous elements in each row an column by small elements. In this way we can also construct configurations with order N degrees of freedom coupled between them starting from the inhomogeneous configurations and changing a finite number of small elements in each row and column by homogeneous configurations. Therefore with this trick

15

one can generate the same number of coupled and uncoupled configurations.

We can also generate new configurations with N^2 degrees of freedom starting from the homogeneous configurations and adding order N large elements. For instance a unitary matrix with N large elements and $N^2 - N$ homogeneous elements. But with this trick we can also generate inhomogeneous configurations starting from inhomogeneous configurations: if we add order N large elements to a matrix with order N large elements the result will be a matrix with order N large elements.

We can generate new configurations with order N^2 degrees of freedom from homogeneous configurations changing a complete row or column of homogeneous elements by small elements. If we apply this trick to an inhomogeneous configuration then the new configuration will have N large elements and N homogeneous elements and these will be coupled to the large elements. But the number of degrees of freedom will be also order N.

All the configurations generated from homogeneous and inhomogeneous configurations with combinations of the above tricks exhaust the set of unitary matrices. Therefore for each configuration with N^2 degrees of freedom generate from some homogeneous configurations we can generate a configuration with N degrees of freedom from an inhomogeneous configuration. Hence the partition function in the large N limit can be split into two terms:

$$Z = A_1 Z_{uncoupled} + Z_{coupled} \qquad (57)$$

where $Z_{uncoupled}$ is the partition function restricted over the set of unitary matrices with only order N nonzero elements. Actually

$$Z_{uncoupled} = N_{deg} \exp\{-NTr(U^\dagger \Lambda U A\} \qquad (58)$$

16

where U is a solution of (56) and N_{deg} is the number of solutions of (56). The partition function $Z_{coupled}$ is bounded by

$$Z_{coupled} \leq Vol \exp\{-NTr(U_S^\dagger \Lambda U_S A\}$$ (59)

where U_S is the absolute minimum in the subset of unitary matrices with N^2 degrees of freedom. And Vol is of the same order as A_1 in the large N limit.

Hence if the absolute minimum of the potential which is given by (56) belongs to the subset of configuration with N degrees of freedom then the partition function $Z_{uncoupled}$ gives the leading contribution to the partition function in the large N limit.

Let us study now the solutions of (56). There are several cases: If the spectrum of the matrix A is non degenerate, then the matrix $U^\dagger \Lambda U$ must be a diagonal matrix because only a diagonal matrix can commute with a diagonal nondegenerate matrix. And if the spectrum of Λ is non degenerate, then the identity matrix is the only solutions of (56). In this case the value of the matrix potential in (50) is $Tr(A\Lambda)$. If the spectrum of Λ is degenerate, then there are an infinite number of unitary matrices which satisfy $U^\dagger \Lambda U = \Lambda$: all the unitary matrices corresponding to change of basis that leave invariant the subspace of eigenvectors of Φ with the same eigenvalue. These unitary matrices are block diagonal, the dimension of each block is given by the dimension of each subspace. Then if these dimensions are order N then there are solutions of (56) that do not belong to the subset of inhomogeneous configurations and the reduction of the number of degrees of freedom does not take place. But this does not hold because the Van der Monde determinant, which arise in the change of variables (23), forbids configurations where the matrix Λ is degenerate. In the large N limit the degeneration of Λ cannot

17

be order N and the solutions of (56) belong to the subset of inhomogeneous configurations.

If the spectrum of A is degenerate, then the matrix $U^\dagger \Lambda U$ must be block diagonal. For instance if

$$
\begin{aligned}
A_{ii} &= a_1 \quad i = 1, \cdots, n_1 \\
A_{ii} &= a_2 \quad i = n_1, \cdots, n_2 \\
&\quad . \\
&\quad . \\
A_{ii} &= a_j \quad i = n_{j-1}, \cdots, N
\end{aligned}
\tag{60}
$$

then $U^\dagger \Lambda U$ is given by j boxes in the diagonal of dimensions given by the set of numbers n_k. In this case the matrix U is also block diagonal and the dimension of each block depends on both matrices A and Λ. If the degeneration of A is order N then there are solutions of (56) which do not belong to the subset of inhomogeneous configurations, but we can chose the external matrix A to be nondegenerate.

In the large N limit the eigenvalue configurations are given by continuum functions $a(x)$ and $\Lambda(x)$. Therefore near a given eigenvalue there are an infinite number of eigenvalues, but this number is order less than N. Therefore the solutions of (56) are unitary matrices which belong to the subset of inhomogeneous configurations in the large N limit and the leading contribution to the partition function is given by Z_I.

All the solutions of (56) give the same contribution to the partition function. Actually the value of the matrix potential for solutions of (56) is always $Tr(A\Lambda)$. And one expect that the number of solutions must be a continuum

18

functional of the eigenvalue configurations. Hence, in the large N limit the partition function is

$$N_S[\lambda(x), a(x)] \exp\{Tr(\Lambda A)\} \qquad (61)$$

where N_S is the number of solutions of (56). It is not possible to find the functional N_S from the equation (56). But the integration in (13) can be performed exactly for finite N[3]. Therefore, from the exact solution is possible to extract the functional N_S in the large N limit.

Let us remark a few things about the equation (56). In the application of one hermitian matrix models to condensed matter, mesoscopic and nuclear physics the matrix is identify with a Hamiltonian and the partition function is a sum over Hamiltonians which satisfy some symmetry[5]. For instance matrix models with real matrices are related to models with time-reversal and spin-rotation symmetries, complex matrices are related to models with time-reversal symmetry broken by a magnetic field or magnetic impurities, and quaternionic matrices are related to models with the spin-rotation symmetry broken by strong spin-orbit scattering for instance. In all this cases the correlation between the eigenvalues $\{\lambda_i\}$ are given by

$$\prod_{i \neq j}(\lambda_i - \lambda_j)^\beta \qquad (62)$$

where β can be 1, 2 or 4 depending on the symmetry of the model. Actually β is the number of degrees of freedom in the matrix elements: 1 for real matrices, 2 for complex matrices and 4 for quaternionic matrices. But it is well known that if one introduce into the matrix potential a coupling with an external matrix, as in (10), the exponent in (62) changes. But let us observe that, in the large N limit, the coupling with an external matrix restricts

19

the sum over all hermitian matrices to the sum over hermitian matrices that commute with the external matrix A (56). Therefore the change in the index β is still related with a change in the symmetry of the set of matrices over which the path integral is defined.

Let us perform the calculation of

$$Z = \int D\Phi \exp\left\{-NTr(\Phi A\Phi A + V(\Phi))\right\} \tag{63}$$

Now the integral over the unitary matrices U is given by:

$$\tilde{Z} = \int DU \exp\left\{-NTr(U^\dagger \Lambda U A U^\dagger \Lambda U A)\right\} \tag{64}$$

One can prove that the classical configurations of the matrix potential are given by

$$(U^\dagger \Lambda U A)^2 = (A U^\dagger \Lambda U)^2. \tag{65}$$

The solutions of (56) are also solutions of the above equations. But there are other solutions. For instance, the configurations which satisfy

$$U^\dagger \Lambda U A = -A U^\dagger \Lambda U \tag{66}$$

are also solutions of the new saddle-point equation. In this case the number of solutions depends on the diagonal matrix A and Λ. For instance, if A is the identity matrix there are not solutions of (66). Actually, there are solutions of (66) only if:

$$a_{ii} = -a_{jj} \tag{67}$$

20

for some i and j. Therefore the form of the matrix A must be:

$$
\begin{pmatrix}
A_1 & & & & & & \\
& -A_1 & & & & & \\
& & A_2 & & & & \\
& & & -A_2 & & & \\
& & & & \cdot & & \\
& & & & & \cdot & \\
& & & & & & D
\end{pmatrix}
\tag{68}
$$

where A_i are matrices proportional to the identity matrix and D is a matrix which does not verify (67). Then the form of the matrix $U^\dagger \Lambda U$ must be

$$
\begin{pmatrix}
0 & B_1 & & & & \\
B_1^\dagger & 0 & & & & \\
& & 0 & B_2 & & \\
& & B_2^\dagger & 0 & & \\
& & & & \cdot & \\
& & & & & \cdot \\
& & & & & & 0
\end{pmatrix}
\tag{69}
$$

where B_i is a complex matrix, and the square root of the eigenvalues of the hermitian matrices $B_i^\dagger B_i$ are the eigenvalues of Λ. The value of the matrix potential for these configurations is

$$
-2\sum_i a_i^2 Tr(B_i^\dagger B_i) \neq Tr(A^2\Lambda^2),
\tag{70}
$$

where $A_i = a_i I$. Whereas for the same matrix A and Λ, the value of the action for the solution of (56) is

$$
2\sum_i a_i Tr(B_i^\dagger B_i) = Tr(A^2\Lambda^2).
\tag{71}
$$

Therefore, solutions of

$$U^\dagger \Lambda U A = A U^\dagger \Lambda U$$
$$U^\dagger \Lambda U A = -A U^\dagger \Lambda U \tag{72}$$

are saddle-point configurations of the partition function (13). But their contributions to the partition function are different. Therefore, we can think the two set of solutions as different vacua of the model. There are other saddle-point configurations: configurations for which the matrix $U^\dagger \Lambda U A$ has two blocks in the diagonal, in such a way that one of the block satisfy the commutation conditions and the other the anticommutation conditions. These sets of solutions give different contribution to the integration over the U matrix.

This is an interesting difference between the linear coupling to an external matrix and the gaussian coupling: the former has only one vacuum whereas in the gaussian coupling there are several vacua of the saddle point equation corresponding to the integration of the unitary matrix. But in all this cases if the degeneration of the external matrix is order one in the large N limit then the solutions of saddle point equation are given by inhomogeneous configurations and the reduction of degrees of freedom holds. If the external matrix A does not verify (67) then the only solutions are given by solutions of

$$U^\dagger \Lambda U A = A U^\dagger \Lambda U \tag{73}$$

In this case the eigenvalues must be all of the same sign because if for some i and j, the corresponding eigenvalues verify $\lambda_i = -\lambda_j$ then there are solutions different from (**??**. This is because the symmetry between A and Λ in the

22

equations. For this vacuum the degeneration factor $N_S[\Lambda, A]$ must the same as in the linear case. Therefore the partition function (13), in the large N limit, becomes:

$$\tilde{Z} \propto \frac{1}{\Delta(\Lambda)} \exp\{-NTr(A^2\Lambda^2)\} \qquad (74)$$

Even though this model can be used to define pure Quantum Gravity with extrinsic curvature its solution is not very interesting from a physical point of view[7, 9].

4. Conclusions

The main conclusion of this paper is that for matrix models with N^2 degrees of freedom for finite N there is a reduction of the number of degrees of freedom in the large N limit, actually the number of degrees of freedom becomes order N. This reduction is the origin of the factorization of the vacuum expectation values of products of observables in the path integral approach in the matrix models studied in this paper.

The reduction showed in this paper is different from the well known Eguchi-Kaway reduction[10]. The EK reduction is a reduction of the external space: the model reduces to a zero dimensional model; while in the reduction studied in this paper the reduction takes place in the internal space of degrees of freedom: actually the matrix models studied in this paper are zero dimensional.

It seems that this reduction is more fundamental than the EK reduction. This reduction is the origin of the factorization of product of observables and the factorization is the origin of the EK reduction.

This result holds only if at least order N^2 degrees of freedom can be thought as the elements of some unitary matrices and the absolute minimum

of the matrix potential, as a function of this unitary matrices, is given by a configuration with only order N degrees of freedom. The first condition is accomplished by every matrix model and also by gauge theories in the lattice because every hermitian or unitary matrix can be split into the diagonal matrix of its eigenvalues and a unitary matrix:$\Phi = U^\dagger \Lambda U$. The second condition seems to depend on the matrix potential. For instance, in two unitary matrix models with matrix potential given by

$$TrV(U_1) + TrV(U_2) + Tr(U_1 U_2) + h.c., \tag{75}$$

there are saddle point configurations which do not verify the second condition. But if one perform the following change of variables

$$U_i \rightarrow \Omega_i^\dagger \theta_i \Omega_i, \tag{76}$$

where θ_i is the diagonal matrix of the eigenvalues of U_i and Ω is a unitary matrix; then the saddle point configurations corresponding to the matrix variable Ω verify the second condition. Actually this problem is analogous to the matrix models studied in this paper. Therefore if for any given matrix model or gauge theory in the lattice is possible to find a change of variables, as for instance (76), such that order N^2 degrees of freedom are the elements of some unitary matrices, for instance Ω in (76), and perhaps there are other N degrees of freedom, for instance the diagonal matrix θ in (76), and the saddle point configurations of the potentials as functions of the former unitary matrices are given by N degrees of freedom only, then the reduction of the number of degrees of freedom holds in the large N limit.

This reduction must be a general result of the large N limit because the factorization of the expectation values of products of observables is a general

24

result and is difficult to understand how the product of observables depending on N^2 degrees of freedom can factorize if fluctuation to all orders enter into the path integral. Therefore we can conjecture that this reduction in the internal space must be a general behaviour of the large N limit in the path integral approach.

References

[1] G. t'Hooft, *Nucl. Phys.***B72** (1974) 461; **B75** (1974) 461.

[2] L. Yaffe, *Rev, Mod. Phys.***54**(1982) 407

[3] C. Itzykson and J. B. Zuber, *J. Math. Phys.* **21**, (1980) 411.

[4] M. L. Metha, *Commun. Math. Phys.* **79** (1981) 327; S. Chanda, G. Mahoux and M. L. Metha *J. Phys.* **A14** (1981) 579; D. Gross and M. J. Newman, *Phys. Lett.* **266B** (1991) 291.

[5] M. L. Metha P. A. Mello, in: Mesoscopic quantum physics, eds., E. Akkermans, G. Montambaux, J. L. Pichard and J. Zinn-Justin, Les Houches, Sesion LXI 1994 (North-Holland, Amsterdam, 1995), p. 435.

[6] J. J. Verbaarschot, *Nucl. Phys. Proc. Suppl.* **B53** (1997) 88.

[7] D. Boulatov and V. Kazakov, *Int. J. Mod. Phys.* **A8** (1993) 809; V. A. Kazakov, M. Staudacher and T. Wynter, *Nucl. Phys.* **B471** (1996) 309.

[8] E. Brézin, C. Itzykson, G. Parisi and J. B. Zuber, *Comm. Math. Phys.* **59**, (1978) 35.

[9] M. J. Bowick, S. M. Catterall, G. Thorleifsson, *Nucl. Phys. Proc. Suppl.* **B53** (1997) 753.

[10] T. Eguchi and H. Kawai, *Phys. Rev. Lett* **48** (1982) 1063; G. Bhanot, U. M Heller and H. Neueberger, *Phys. Lett* **113B** (1982) 47; G. Parisi, *Phys. Lett.* **112B** (1982) 463; S. R. Das and S. Wadia, *Phys. Lett* **117B** (1982) 228; D. J. Ross and Y. Kitazawa, *Nucl. Phys.* **B206** (1982) 440; Y.

Kitazawa and S. R. Wadia, *Phys. Lett.* **120B** (1983) 377; H. Neueberger, *Phys. Lett.* **119B** (1982) 170; T. Eguchi and R. Nakayama, *Phys. Lett.* **122B** (1983) 59; A. González-Arroyo and M. Okawa, *Phys. Rev.* **D27** (1983) 2397.